

CLASSIFICATION OF EXTREMAL ELLIPTIC $K3$ SURFACES AND FUNDAMENTAL GROUPS OF OPEN $K3$ SURFACES

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ABSTRACT. We present a complete list of extremal elliptic $K3$ surfaces (Theorem 1.1). As an application, we give a sufficient condition for the topological fundamental group of complement to an ADE -configuration of smooth rational curves on a $K3$ surface to be trivial (Proposition 4.1 and Theorem 4.3).

1. INTRODUCTION

A complex elliptic $K3$ surface $f : X \rightarrow \mathbb{P}^1$ with a section O is said to be *extremal* if the Picard number $\rho(X)$ of X is 20 and the Mordell-Weil group MW_f of f is finite. The purpose of this paper is to present the complete list of all extremal elliptic $K3$ surfaces. As an application, we show that, if an ADE -configuration of smooth rational curves on a $K3$ surface satisfies a certain condition, then the topological fundamental group of the complement is trivial. (See Theorem 4.3 for the precise statement.)

Let $f : X \rightarrow \mathbb{P}^1$ be an elliptic $K3$ surface with a section O . We denote by R_f the set of all points $v \in \mathbb{P}^1$ such that $f^{-1}(v)$ is reducible. For a point $v \in R_f$, let $f^{-1}(v)^\#$ be the union of irreducible components of $f^{-1}(v)$ that are disjoint from the zero section O . It is known that the cohomology classes of irreducible components of $f^{-1}(v)^\#$ form a negative definite root lattice $S_{f,v}$ of type A_l , D_m or E_n in $H^2(X; \mathbb{Z})$. Let $\tau(S_{f,v})$ be the type of this lattice. We define Σ_f to be the formal sum of these types;

$$\Sigma_f := \sum_{v \in R_f} \tau(S_{f,v}).$$

The Néron-Severi lattice NS_X of X is defined to be $H^{1,1}(X) \cap H^2(X; \mathbb{Z})$, and the transcendental lattice T_X of X is defined to be the orthogonal complement of NS_X in $H^2(X; \mathbb{Z})$. We call the triple (Σ_f, MW_f, T_X) the *data* of the elliptic $K3$ surface $f : X \rightarrow \mathbb{P}^1$. When $f : X \rightarrow \mathbb{P}^1$ is extremal, the transcendental lattice T_X is a positive definite even lattice of rank 2.

Theorem 1.1. *There exists an extremal elliptic $K3$ surface $f : X \rightarrow \mathbb{P}^1$ with data (Σ_f, MW_f, T_X) if and only if (Σ_f, MW_f, T_X) appears in Table 2 given at the end of this paper.*

In Table 2, the transcendental lattice T_X is expressed by the coefficients of its Gram matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

See Subsection 2.1 on how to recover the $K3$ surface X from T_X .

The classification of *semi-stable* extremal elliptic $K3$ surfaces has been done by Miranda and Persson [7] and complemented by Artal-Bartolo, Tokunaga and Zhang [1]. We can check that the semi-stable part of our list (No. 1- No. 112) coincides with theirs. Nishiyama [12] classified all elliptic fibrations (not necessarily extremal) on certain $K3$ surfaces. On the other hand, Ye [19] has independently classified all extremal elliptic $K3$ surfaces with no semi-stable singular fibers by different methods from ours.

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2. PRELIMINARIES

2.1. Transcendental lattice of singular $K3$ surfaces. Let \mathcal{Q} be the set of symmetric matrices

$$Q = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

of integer coefficients such that a and c are even and that the corresponding quadratic forms are positive definite. The group $GL_2(\mathbb{Z})$ acts on \mathcal{Q} from right by

$$Q \mapsto {}^t g \cdot Q \cdot g,$$

where $g \in GL_2(\mathbb{Z})$. Let Q_1 and Q_2 be two matrices in \mathcal{Q} , and let L_1 and L_2 be the positive definite even lattices of rank 2 whose Gram matrices are Q_1 and Q_2 , respectively. Then L_1 and L_2 are isomorphic as lattices if and only if Q_1 and Q_2 are in the same orbit under the action of $GL_2(\mathbb{Z})$. On the other hand, each orbit in \mathcal{Q} under the action of $SL_2(\mathbb{Z})$ contains a unique matrix with coefficients satisfying

$$-a < 2b \leq a \leq c, \quad \text{with } b \geq 0 \text{ if } a = c.$$

(See, for example, Conway and Sloane [3, p. 358].) Hence each orbit in \mathcal{Q} under the action of $GL_2(\mathbb{Z})$ contains a unique matrix with coefficients satisfying

$$(2.1) \quad 0 \leq 2b \leq a \leq c.$$

In Table 2, the transcendental lattice is represented by the Gram matrix satisfying the condition (2.1).

Let X be a $K3$ surface with $\rho(X) = 20$; that is, X is a singular $K3$ surface in the terminology of Shioda and Inose [16]. The transcendental lattice T_X can be naturally oriented by means of a holomorphic two form on X (cf. [16, p. 128]). Let \mathcal{S} denote the set of isomorphism classes of singular $K3$ surfaces. Using the natural orientation on the transcendental lattice, we can lift the map $\mathcal{S} \rightarrow \mathcal{Q}/GL_2(\mathbb{Z})$ given by $X \mapsto T_X$ to the map $\mathcal{S} \rightarrow \mathcal{Q}/SL_2(\mathbb{Z})$.

Proposition 2.1 (Shioda and Inose [16]). *This map $\mathcal{S} \rightarrow \mathcal{Q}/SL_2(\mathbb{Z})$ is bijective.* \square

Moreover, Shioda and Inose [16] gave us a method to construct explicitly the singular $K3$ surface corresponding to a given element of $\mathcal{Q}/SL_2(\mathbb{Z})$ by means of Kummer surfaces. The injectivity of the map $\mathcal{S} \rightarrow \mathcal{Q}/SL_2(\mathbb{Z})$ had been proved by Piatetskii-Shapiro and Shafarevich [14].

Suppose that an orbit $[Q] \in \mathcal{Q}/GL_2(\mathbb{Z})$ is represented by a matrix Q satisfying (2.1). Let $\rho : \mathcal{Q}/SL_2(\mathbb{Z}) \rightarrow \mathcal{Q}/GL_2(\mathbb{Z})$ be the natural projection. Then we

have

$$|\rho^{-1}([Q])| = \begin{cases} 2 & \text{if } 0 < 2b < a < c \\ 1 & \text{otherwise.} \end{cases}$$

Therefore, if a data in Table 2 satisfies $a = c$ or $b = 0$ or $2b = a$ (resp. $0 < 2b < a < c$), then the number of the isomorphism classes of $K3$ surfaces that possess a structure of the extremal elliptic $K3$ surfaces with the given data is one (resp. two).

2.2. Roots of a negative definite even lattice. Let M be a negative definite even lattice. A vector of M is said to be a *root* of M if its norm is -2 . We denote by $\text{root}(M)$ the number of roots of M , and by M_{root} the sublattice of M generated by the roots of M . Suppose that a Gram matrix (a_{ij}) of M is given. Then $\text{root}(M)$ can be calculated by the following method. Let

$$g_r(x) = - \sum_{i,j=1}^r a_{ij} x_i x_j$$

be the positive definite quadratic form associated with the opposite lattice M^- of M , where r is the rank of M . We consider the bounded closed subset

$$E(g_r, 2) := \{x \in \mathbb{R}^r ; g_r(x) \leq 2\}$$

of \mathbb{R}^r . Then we have

$$\text{root}(M) + 1 = |E(g_r, 2) \cap \mathbb{Z}^r|,$$

where $+1$ comes from the origin. For a positive integer k less than r , we write by $p_k : \mathbb{R}^r \rightarrow \mathbb{R}^k$ the projection $(x_1, \dots, x_r) \mapsto (x_1, \dots, x_k)$. Then there exist a positive definite quadratic form g_k of variables (x_1, \dots, x_k) and a positive real number σ_k such that

$$p_k(E(g_r, 2)) = E(g_k, \sigma_k) := \{y \in \mathbb{R}^k ; g_k(y) \leq \sigma_k\}.$$

The projection $(x_1, \dots, x_{k+1}) \mapsto (x_1, \dots, x_k)$ maps $E(g_{k+1}, \sigma_{k+1})$ to $E(g_k, \sigma_k)$. Hence, if we have the list of the points of $E(g_k, \sigma_k) \cap \mathbb{Z}^k$, then it is easy to make the list of the points of $E(g_{k+1}, \sigma_{k+1}) \cap \mathbb{Z}^{k+1}$. Thus, starting from $E(g_1, \sigma_1) \cap \mathbb{Z}$, we can make the list of the points of $E(g_r, 2) \cap \mathbb{Z}^r$ by induction on k .

2.3. Root lattices of type ADE . A *root type* is, by definition, a finite formal sum Σ of A_l , D_m and E_n with non-negative integer coefficients;

$$\Sigma = \sum_{l \geq 1} a_l A_l + \sum_{m \geq 4} d_m D_m + \sum_{n=6}^8 e_n E_n.$$

We denote by $L(\Sigma)$ the negative definite root lattice corresponding to Σ . The rank of $L(\Sigma)$ is given by

$$\text{rank}(L(\Sigma)) = \sum_{l \geq 1} a_l l + \sum_{m \geq 4} d_m m + \sum_{n=6}^8 e_n n,$$

and the number of roots of $L(\Sigma)$ is given by

$$(2.2) \quad \text{root}(L(\Sigma)) = \sum_{l \geq 1} a_l (l^2 + l) + \sum_{m \geq 4} d_m (2m^2 - 2m) + 72e_6 + 126e_7 + 240e_8.$$

(See, for example, Bourbaki [2].) Because of $L(\Sigma)_{root} = L(\Sigma)$, we have

$$(2.3) \quad L(\Sigma_1) \cong L(\Sigma_2) \iff \Sigma_1 = \Sigma_2.$$

We also define $eu(\Sigma)$ by

$$eu(\Sigma) := \sum_{l \geq 1} a_l(l+1) + \sum_{m \geq 4} d_m(m+2) + \sum_{n=6}^8 e_n(n+2).$$

Lemma 2.2. *Let $f : X \rightarrow \mathbb{P}^1$ be an elliptic K3 surface. Then $eu(\Sigma_f)$ is at most 24. Moreover, if $eu(\Sigma_f) < 24$, then there exists at least one singular fiber of type I_1 , II, III or IV.*

Proof. Let $e(Y)$ denote the topological euler number of a CW -complex Y . Then $e(X) = 24$ is equal with the sum of topological euler numbers of singular fibers of f . Every singular fiber has a positive topological euler number. We have defined $eu(\Sigma)$ in such a way that, if $v \in R_f$, then $eu(\tau(S_{f,v})) \leq e(f^{-1}(v))$ holds, and if $eu(\tau(S_{f,v})) < e(f^{-1}(v))$, then the type of the fiber $f^{-1}(v)$ is either III or IV. Hence $eu(\Sigma_f)$ does not exceed the sum of the topological euler numbers of reducible singular fibers, and if $eu(\Sigma_f) < 24$, then there is an irreducible singular fiber or a singular fiber of type III or IV. \square

2.4. Discriminant form and overlattices. Let L be an even lattice, L^\vee the dual of L , D_L the discriminant group L^\vee/L of L , and q_L the discriminant form on D_L . (See Nikulin [11, n. 4] for the definitions.) An overlattice of L is, by definition, an integral sublattice of the \mathbb{Q} -lattice L^\vee containing L .

Lemma 2.3 (Nikulin [11] Proposition 1.4.2). (1) *Let A be an isotopic subgroup of (D_L, q_L) . Then the pre-image $M := \phi_L^{-1}(A)$ of A by the natural projection $\phi_L : L^\vee \rightarrow D_L$ is an overlattice of L , and the discriminant form (D_M, q_M) of M is isomorphic to $(A^\perp/A, q_L|_{A^\perp/A})$, where A^\perp is the orthogonal complement of A in D_L , and $q_L|_{A^\perp/A}$ is the restriction of q_L to A^\perp/A . (2) *The correspondence $A \mapsto M$ gives a bijection from the set of isotopic subgroups of (D_L, q_L) to the set of even overlattices of L .* \square*

Lemma 2.4 (Nikulin [11] Corollary 1.6.2). *Let S and K be two even lattices. Then the following two conditions are equivalent. (i) There is an isomorphism $\gamma : D_S \xrightarrow{\sim} D_K$ of abelian groups such that $\gamma^*q_K = -q_S$. (ii) There is an even unimodular overlattice of $S \oplus K$ into which S and K are primitively embedded.* \square

2.5. Néron-Severi groups of elliptic K3 surfaces. Let $f : X \rightarrow \mathbb{P}^1$ be an elliptic K3 surface with the zero section O . In the Néron-Severi lattice NS_X of X , the cohomology classes of the zero section O and a general fiber of f generate a sublattice U_f of rank 2, which is isomorphic to the hyperbolic lattice

$$H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let W_f be the orthogonal complement of U_f in NS_X . Because U_f is unimodular, we have $NS_X = U_f \oplus W_f$. Because U_f is of signature $(1, 1)$ and NS_X is of signature $(1, \rho(X) - 1)$, W_f is negative definite of rank $\rho(X) - 2$. Note that W_f contains the sublattice

$$S_f := \bigoplus_{v \in R_f} S_{f,v}$$

generated by the cohomology classes of irreducible components of reducible fibers of f that are disjoint from the zero section. By definition, S_f is isomorphic to $L(\Sigma_f)$.

Lemma 2.5 (Nishiyama [12] Lemma 6.1). *The sublattice S_f of W_f coincides with $(W_f)_{\text{root}}$, and the Mordell-Weil group MW_f of f is isomorphic to W_f/S_f . In particular, $\text{root}(L(\Sigma_f))$ is equal with $\text{root}(W_f)$. \square*

Because $W_f \oplus U_f \oplus T_X$ has an even unimodular overlattice $H^2(X; \mathbb{Z})$ into which $NS_X = W_f \oplus U_f$ and T_X are primitively embedded, and because the discriminant form of NS_X is equal with the discriminant form of W_f by $D_{U_f} = (0)$, Lemma 2.4 implies the following:

Corollary 2.6. *There is an isomorphism $\gamma : D_{W_f} \xrightarrow{\sim} D_{T_X}$ of abelian groups such that $\gamma^* q_{T_X}$ coincides with $-q_{W_f}$. \square*

2.6. Existence of elliptic K3 surfaces. Let Λ be the K3 lattice $L(2E_8) \oplus H^{\oplus 3}$.

Lemma 2.7 (Kondō [5] Lemma 2.1). *Let T be a positive definite primitive sublattice of Λ with $\text{rank}(T) = 2$, and T^\perp the orthogonal complement of T in Λ . Suppose that T^\perp contains a sublattice H_T isomorphic to the hyperbolic lattice. Let M_T be the orthogonal complement of H_T in T^\perp . Then there exist an elliptic K3 surface $f : X \rightarrow \mathbb{P}^1$ such that $T_X \cong T$ and $W_f \cong M_T$.*

Proof. By the surjectivity of the period map of the moduli of K3 surfaces (cf. Todorov [17]), there exist a K3 surface X and an isomorphism $\alpha : H^2(X; \mathbb{Z}) \cong \Lambda$ of lattices such that $\alpha^{-1}(T) = T_X$. By Kondō [5, Lemma 2.1], the K3 surface X has an elliptic fibration $f : X \rightarrow \mathbb{P}^1$ with a section such that $\mathbb{Z}[F]^\perp / \mathbb{Z}[F] \cong M_T$, where $[F] \in U_f$ is the cohomology class of a fiber of f , and $\mathbb{Z}[F]^\perp$ is the orthogonal complement of $[F]$ in the Néron-Severi lattice NS_X . Because NS_X is equal with $U_f \oplus W_f$, and because $\mathbb{Z}[F]^\perp \cap U_f$ coincides with $\mathbb{Z}[F]$, we see that $\mathbb{Z}[F]^\perp / \mathbb{Z}[F]$ is isomorphic to W_f . \square

2.7. Datum of extremal elliptic K3 surfaces.

Proposition 2.8. *A triple (Σ, MW, T) consisting of a root type Σ , a finite abelian group MW and a positive definite even lattice T of rank 2 is a data of an extremal elliptic K3 surface if and only if the following hold:*

- (D1) $\text{length}(MW) \leq 2$, $\text{rank}(L(\Sigma)) = 18$ and $eu(\Sigma) \leq 24$.
- (D2) *There exists an overlattice M of $L(\Sigma)$ satisfying the following:*
 - (D2-a) $M/L(\Sigma) \cong MW$,
 - (D2-b) *there exists an isomorphism $\gamma : D_M \xrightarrow{\sim} D_T$ of abelian groups such that $\gamma^* q_T = -q_M$, and*
 - (D2-c) $\text{root}(L(\Sigma)) = \text{root}(M)$.

Proof. Suppose that there exists an extremal elliptic K3 surface $f : X \rightarrow \mathbb{P}^1$ with data equal with (Σ, MW, T) . It is obvious that Σ and MW satisfies the condition (D1). Via the isomorphism $S_f \cong L(\Sigma)$, the overlattice W_f of S_f corresponds to an overlattice M of $L(\Sigma)$, which satisfies the conditions (D2-a)-(D2-c) by Lemma 2.5 and Corollary 2.6. Conversely, suppose that (Σ, MW, T) satisfies the conditions (D1) and (D2). By Lemma 2.4, the condition (D2-b) and $D_H = 0$ imply that there exists an even unimodular overlattice of $M \oplus H \oplus T$ into which $M \oplus H$ and T are primitively embedded. By the theorem of Milnor (see, for example, Serre [15]) on the classification of even unimodular lattices, any even unimodular lattice of

signature $(3, 19)$ is isomorphic to the $K3$ lattice Λ . Then Lemma 2.7 implies that there exists an elliptic $K3$ surface $f : X \rightarrow \mathbb{P}^1$ satisfying $W_f \cong M$ and $T_X \cong T$. The condition $(D2 - c)$ implies $M_{root} = L(\Sigma)$. Combining this with Lemma 2.5, we see that $S_f \cong L(\Sigma)$. Then (2.2) implies that $\Sigma_f = \Sigma$. Using Lemma 2.5 and the condition $(D2 - a)$, we see that $MW_f \cong MW$. Thus the data of $f : X \rightarrow \mathbb{P}^1$ coincides with (Σ, MW, T) . \square

Remark 2.9. In the light of Lemma 2.3, the condition $(D2)$ is equivalent to the following:

- (D3) There exists an isotopic subgroup A of $(D_{L(\Sigma)}, q_{L(\Sigma)})$ satisfying the following:
 - (D3 - a) A is isomorphic to MW ,
 - (D3 - b) there exists an isomorphism $\gamma : A^\perp/A \xrightarrow{\sim} D_T$ of abelian groups such that $\gamma^* q_T = -q_{L(\Sigma)}|_{A^\perp/A}$, and
 - (D3 - c) $\text{root}(\phi_{L(\Sigma)}^{-1}(A))$ is equal with $\text{root}(L(\Sigma))$, where $\phi_{L(\Sigma)} : L(\Sigma)^\vee \rightarrow D_{L(\Sigma)}$ is the natural projection.

Remark 2.10. We did not use the conditions $\text{length}(MW) \leq 2$ and $eu(\Sigma) \leq 24$ in the proof of the “if” part of Proposition 2.8. It follows that, if (Σ, MW, T) satisfies $\text{rank}(L(\Sigma)) = 18$ and the condition $(D2)$, then $\text{length}(MW) \leq 2$ and $eu(\Sigma) \leq 24$ follow automatically. This fact can be used when we check the computer program described in the next section.

3. MAKING THE LIST

First we list up all root types Σ satisfying $\text{rank}(L(\Sigma)) = 18$ and $eu(\Sigma) \leq 24$. This list \mathcal{L} consists of 712 elements.

Next we run a program that takes an element Σ of the list \mathcal{L} as an input and proceeds as follows.

Step 1. The program calculates the intersection matrix of $L(\Sigma)^\vee$. Using this matrix, it calculates the discriminant form of $L(\Sigma)$, and decomposes it into p -parts;

$$(D_{L(\Sigma)}, q_{L(\Sigma)}) = \bigoplus_p (D_{L(\Sigma)}, q_{L(\Sigma)})_p,$$

where p runs through the set $\{p_1, \dots, p_k\}$ of prime divisors of the discriminant $|D_{L(\Sigma)}|$ of $L(\Sigma)$. We write the p_i -part of $(D_{L(\Sigma)}, q_{L(\Sigma)})$ by $(D_{L(\Sigma), i}, q_{L(\Sigma), i})$.

Step 2. For each p_i , it calculates the set $I(p_i)$ of all pairs (A, A^\perp) of an isotopic subgroup A of $(D_{L(\Sigma), i}, q_{L(\Sigma), i})$ and its orthogonal complement A^\perp such that $\text{length}(A) \leq 2$.

Step 3. For each element

$$\mathcal{A} := ((A_1, A_1^\perp), \dots, (A_k, A_k^\perp)) \in I(p_1) \times \dots \times I(p_k),$$

it calculates the $\mathbb{Q}/2\mathbb{Z}$ -valued quadratic form

$$q_{\mathcal{A}} := q_{L(\Sigma), 1}|_{A_1^\perp/A_1} \times \dots \times q_{L(\Sigma), k}|_{A_k^\perp/A_k}$$

on the finite abelian group

$$D_{\mathcal{A}} := A_1^\perp/A_1 \times \dots \times A_k^\perp/A_k.$$

Let $d(\mathcal{A})$ be the order of $D_{\mathcal{A}}$.

Step 4. It generates the list $\mathcal{T}(d(\mathcal{A}))$ of positive definite even lattices of rank 2 with discriminant equal with $d(\mathcal{A})$. For each $T \in \mathcal{T}(d(\mathcal{A}))$, it calculates the discriminant form of T and decomposes it into p -parts. If D_T is isomorphic to $D_{\mathcal{A}}$ and q_T is isomorphic to $-q_{\mathcal{A}}$, then it proceeds to the next step. Note that the automorphism group of a finite abelian p -group of length ≤ 2 is easily calculated, and hence it is an easy task to check whether two given quadratic forms on the finite abelian p -group of length ≤ 2 are isomorphic or not.

Step 5. It calculates the Gram matrix of the sublattice $\tilde{L}(\mathcal{A})$ of $L(\Sigma)^\vee$ generated by $L(\Sigma) \subset L(\Sigma)^\vee$ and the pull-backs of generators of the subgroups $A_i \subset D_{L(\Sigma),i}$ by the projection $L(\Sigma)^\vee \rightarrow D_{L(\Sigma)} \rightarrow D_{L(\Sigma),i}$. Then it calculates $\text{root}(\tilde{L}(\mathcal{A}))$ by the method described in the subsection 2.2. If $\text{root}(\tilde{L}(\mathcal{A}))$ is equal with $\text{root}(L(\Sigma))$ calculated by (2.2), then it puts out the pair of the finite abelian group

$$MW := A_1 \times \cdots \times A_k$$

and the lattice T .

Then (Σ, MW, T) satisfies the conditions (D1) and (D3), and all triples (Σ, MW, T) satisfying (D1) and (D3) are obtained by this program.

4. FUNDAMENTAL GROUPS OF OPEN K3 SURFACES

A simple normal crossing divisor Δ on a K3 surface X is said to be an *ADE-configuration of smooth rational curves* if each irreducible component of Δ is a smooth rational curve and the intersection matrix of the irreducible components of Δ is a direct sum of the Cartan matrices of type A_l , D_m or E_n multiplied by -1 . It is known that Δ is an ADE-configuration of smooth rational curves if and only if each connected component of Δ can be contracted to a rational double point. We consider the following quite plausible hypothesis. Let Δ be an ADE-configuration of smooth rational curves on a K3 surface X .

Hypothesis. If $\pi_1^{\text{alg}}(X \setminus \Delta)$ is trivial, then so is $\pi_1(X \setminus \Delta)$.

Here $\pi_1^{\text{alg}}(X \setminus \Delta)$ is the algebraic fundamental group of $X \setminus \Delta$, which is the pro-finite completion of the topological fundamental group $\pi_1(X \setminus \Delta)$.

Proposition 4.1. *Suppose that Hypothesis is true for any ADE-configuration of smooth rational curves on an arbitrary K3 surface. Let Δ be an ADE-configuration of smooth rational curves on a K3 surface X . Then $\pi_1(X \setminus \Delta)$ satisfies one of the following:*

- (i) $\pi_1(X \setminus \Delta)$ is trivial.
- (ii) There exist a complex torus T of dimension 2 and a finite automorphism group G of T such that T/G is birational to X and that $\pi_1(X \setminus \Delta)$ fits in the exact sequence

$$1 \longrightarrow \pi_1(T) \longrightarrow \pi_1(X \setminus \Delta) \longrightarrow G \longrightarrow 1.$$

- (iii) $\pi_1(X \setminus \Delta)$ is isomorphic to a symplectic automorphism group of a K3 surface.

Remark 4.2. Fujiki [4] classified the automorphism groups of complex tori of dimension 2. In particular, the G in (ii) is either one of $\mathbb{Z}/(n)$ ($n = 2, 3, 4, 6$), Q_8 (Quaternion of order 8), D_{12} (Dihedral of order 12) and T_{24} (Tetrahedral of order

24), whence the $\pi_1(X \setminus \Delta)$ in (ii) is a soluble group. Mukai [9] presented the complete list of symplectic automorphism groups of $K3$ surfaces. (See also Kondō [6] and Xiao [18].) Under Hypothesis, therefore, we know what groups can appear as $\pi_1(X \setminus \Delta)$.

Proof of Proposition 4.1. Suppose that $\pi_1(X \setminus \Delta)$ is non-trivial. By Hypothesis, $\pi_1^{alg}(X \setminus \Delta)$ is also non-trivial. For a surjective homomorphism $\phi : \pi_1(X \setminus \Delta) \rightarrow G$ from $\pi_1(X \setminus \Delta)$ to a finite group G , we denote by

$$\psi_\phi : \tilde{Y}_\phi \longrightarrow X$$

the finite Galois cover of X corresponding to ϕ , which is étale over $X \setminus \Delta$ and whose Galois group is canonically isomorphic to G . Let $\rho : \tilde{Y}'_\phi \rightarrow \tilde{Y}_\phi$ be the resolution of singularities, and $\gamma : \tilde{Y}'_\phi \rightarrow Y_\phi$ the contraction of (-1) -curves. We denote by Δ_ϕ the union of one-dimensional irreducible components of $\gamma(\rho^{-1}(\psi_\phi^{-1}(\Delta)))$. Then it is easy to see that Y_ϕ is either a $K3$ surface or a complex torus of dimension 2, and that the Galois group G of ψ_ϕ acts on Y_ϕ symplectically. Moreover, Δ_ϕ is an empty set or an ADE -configuration of smooth rational curves. We have an exact sequence

$$1 \longrightarrow \pi_1(Y_\phi \setminus \Delta_\phi) \longrightarrow \pi_1(X \setminus \Delta) \longrightarrow G \longrightarrow 1,$$

because $\pi_1(\tilde{Y}_\phi \setminus \psi_\phi^{-1}(\Delta))$ is isomorphic to $\pi_1(Y_\phi \setminus \Delta_\phi)$. Suppose that there exists $\phi : \pi_1(X \setminus \Delta) \rightarrow G$ such that Y_ϕ is a complex torus of dimension 2. Then Δ_ϕ is empty, and hence (ii) occurs. Suppose that no complex tori of dimension 2 appear as a finite Galois cover of X branched in Δ . Then any finite quotient group of $\pi_1(X \setminus \Delta)$ must appear in Mukai's list of symplectic automorphism groups of $K3$ surfaces. Because this list consists of finite number of isomorphism classes of finite groups, there exists a maximal finite quotient $\phi_{max} : \pi_1(X \setminus \Delta) \rightarrow G_{max}$ of $\pi_1(X \setminus \Delta)$. Then $\pi_1(Y_{\phi_{max}} \setminus \Delta_{\phi_{max}})$ has no non-trivial finite quotient group, and hence it is trivial by Hypothesis. Thus (iii) occurs. \square

For an ADE -configuration Δ of smooth rational curves on a $K3$ surface X , we denote by $\mathbb{Z}[\Delta]$ the sublattice of $H^2(X; \mathbb{Z})$ generated by the cohomology classes of the irreducible components of Δ , which is isomorphic to a negative definite root lattice of type ADE . We denote by Σ_Δ the root type such that $\mathbb{Z}[\Delta]$ is isomorphic to $L(\Sigma_\Delta)$. Using the list of extremal elliptic $K3$ surfaces, we prove the following theorem. We first consider the following conditions on a root type Σ (see (2.4) for the definition of $D_{L(\Sigma)}$).

- (N1) $\text{rank}(L(\Sigma)) \leq 18$, and
- (N2) $\text{length}(D_{L(\Sigma)}) \leq 20 - \text{rank}(L(\Sigma))$.

Theorem 4.3. *Let X be a $K3$ surface and Δ an ADE -configuration of smooth rational curves on X . Suppose that the root type Σ_Δ satisfies conditions (N1) and (N2). If $\mathbb{Z}[\Delta]$ is primitive in $H^2(X; \mathbb{Z})$ then $\pi_1(X \setminus \Delta)$ is trivial.*

In virtue of Lemma 4.6 below, we can easily derive the following:

Corollary 4.4. *Let X be a $K3$ surface and Δ an ADE -configuration of smooth rational curves on X . Suppose that Σ_Δ satisfies the conditions (N1) and (N2). Then Hypothesis is true.* \square

Remark 4.5. The conditions (N1) and (N2) come from Nikulin [11, Theorem 1.14.1] (see also Morrison [8, Theorem 2.8]), which gives a sufficient condition for the uniqueness of the primitive embedding of $L(\Sigma)$ into the K3 lattice Λ .

First we prepare some lemmas. Let $\overline{\mathbb{Z}[\Delta]}$ be the primitive closure of $\mathbb{Z}[\Delta]$ in $H^2(X; \mathbb{Z})$.

Lemma 4.6 (Xiao [18] Lemma 2). *The dual of the abelianisation of $\pi_1(X \setminus \Delta)$ is canonically isomorphic to $\overline{\mathbb{Z}[\Delta]}/\mathbb{Z}[\Delta]$. In particular, if $\pi_1^{alg}(X \setminus \Delta)$ is trivial, then $\mathbb{Z}[\Delta]$ is primitive in $H^2(X; \mathbb{Z})$. \square*

Let Γ_1 and Γ_2 be graphs with the set of vertices denoted by $\text{Vert}(\Gamma_1)$ and $\text{Vert}(\Gamma_2)$, respectively. An embedding of Γ_1 into Γ_2 is, by definition, an injection $f : \text{Vert}(\Gamma_1) \rightarrow \text{Vert}(\Gamma_2)$ such that, for any $u, v \in \text{Vert}(\Gamma_1)$, $f(u)$ and $f(v)$ are connected by an edge of Γ_2 if and only if u and v are connected by an edge of Γ_1 .

Let $\Gamma(\Sigma)$ denote the Dynkin graph of Σ .

Lemma 4.7. *Suppose that Σ satisfies the conditions (N1) and (N2). Then there exists Σ' satisfying $\text{rank}(L(\Sigma')) = 18$ and the condition (N2) such that $\Gamma(\Sigma)$ can be embedded in $\Gamma(\Sigma')$.*

Proof. This is checked by listing up all Σ satisfying the conditions (N1) and (N2) using computer. \square

Lemma 4.8. *Let $f : X \rightarrow \mathbb{P}^1$ be an elliptic surface with the zero section O . Suppose that a fiber $f^{-1}(v)$ over $v \in \mathbb{P}^1$ is a singular fiber of type III or IV. Let Ξ be a union of some irreducible components of $f^{-1}(v)$ that does not coincide with the whole fiber $f^{-1}(v)$. If U is a small open disk on \mathbb{P}^1 with the center v , then $f^{-1}(U) \setminus (\Xi \cup (f^{-1}(U) \cap O))$ has an abelian fundamental group.*

Proof. This can be proved easily by the van-Kampen theorem. \square

Lemma 4.9. *Let Σ be satisfying the conditions (N1) and (N2). Suppose that (X, Δ) and (X', Δ') satisfy the following:*

- (a) $\Sigma_\Delta = \Sigma_{\Delta'} = \Sigma$,
- (b) $\overline{\mathbb{Z}[\Delta]} = \mathbb{Z}[\Delta]$ and $\overline{\mathbb{Z}[\Delta']} = \mathbb{Z}[\Delta']$.

Then there exists a connected continuous family (X_t, Δ_t) parameterized by $t \in [0, 1]$ such that $(X_0, \Delta_0) = (X, \Delta)$, $(X_1, \Delta_1) = (X', \Delta')$ and that (X_t, Δ_t) are diffeomorphic to one another. In particular, $\pi_1(X \setminus \Delta)$ is isomorphic to $\pi_1(X' \setminus \Delta')$.

Proof. By Nikulin [11, Theorem 1.14.1], the primitive embedding of $L(\Sigma)$ into the K3 lattice Λ is unique up to $\text{Aut}(\Lambda)$. Hence the assertion follows from Nikulin's connectedness theorem [10, Theorem 2.10]. \square

Proof of Theorem 4.3. Let us consider the following:

Claim 1. Suppose that Σ satisfies $\text{rank}(L(\Sigma)) = 18$ and the condition (N2). Then there exists an ADE-configuration of smooth rational curves Δ_Σ on a K3 surface X_Σ such that $\Sigma_{\Delta_\Sigma} = \Sigma$ and $\pi_1(X_\Sigma \setminus \Delta_\Sigma) = \{1\}$.

We deduce Theorem 4.3 from Claim 1. Suppose that Δ is an ADE-configuration of smooth rational curves on a K3 surface X such that Σ_Δ satisfies the conditions (N1) and (N2), and that $\mathbb{Z}[\Delta]$ is primitive in $H^2(X; \mathbb{Z})$. By Lemma 4.7, there

exists Σ_1 satisfying $\text{rank}(L(\Sigma_1)) = 18$ and the condition (N2) such that $\Gamma(\Sigma_\Delta)$ is embedded into $\Gamma(\Sigma_1)$. By Claim 1, we have (X_1, Δ_1) such that $\Sigma_{\Delta_1} = \Sigma_1$ and $\pi_1(X_1 \setminus \Delta_1) = \{1\}$. Let $\Delta' \subset \Delta_1$ be the sub-configuration of smooth rational curves on X_1 corresponding to the subgraph $\Gamma(\Sigma_\Delta) \hookrightarrow \Gamma(\Sigma_1) = \Gamma(\Sigma_{\Delta_1})$. There is a surjection from $\pi_1(X_1 \setminus \Delta_1)$ to $\pi_1(X_1 \setminus \Delta')$, and hence $\pi_1(X_1 \setminus \Delta')$ is trivial. In particular, $\mathbb{Z}[\Delta']$ is primitive in $H^2(X_1; \mathbb{Z})$. Because of $\Sigma_{\Delta'} = \Sigma_\Delta$, Lemma 4.9 implies that $\pi_1(X \setminus \Delta)$ is isomorphic to $\pi_1(X_1 \setminus \Delta')$. Thus $\pi_1(X \setminus \Delta)$ is trivial.

Let $f : X \rightarrow \mathbb{P}^1$ be an extremal elliptic K3 surface. For a point $v \in R_f$, we denote the total fiber of f over v by

$$\sum_{i=1}^{r_v} m_{v,i} C_{v,i},$$

where $m_{v,i}$ is the multiplicity of the irreducible component $C_{v,i}$ of $f^{-1}(v)$. We denote by Γ_f the union of the zero section and all irreducible fibers $f^{-1}(v)$ ($v \in R_f$).

Claim 2. Suppose that $MW_f = (0)$. Suppose that a sub-configuration Δ of Γ_f satisfies the following two conditions.

(Z1) The number of $v \in R_f$ such that $m_{v,i} = 1 \implies C_{v,i} \subset \Delta$ holds is at most one.
 (Z2) Either one of the following holds:

(Z2-a) The configuration Δ does not contain the zero section,

(Z2-b) there is a point $v_1 \in R_f$ such that the type $\tau(S_{f,v_1})$ is A_1 and that $F_1 := f^{-1}(v_1)$ and Δ have no common irreducible components, or

(Z2-c) $eu(\Sigma_f) \leq 23$.

Then $\pi_1(X \setminus \Delta)$ is trivial.

Proof of Claim 2. By Lemma 2.5, the assumption $MW_f = (0)$ implies that the cohomology classes $[O]$ and $[C_{v,i}]$ ($v \in R_f, i = 1, \dots, r_v$) of the irreducible components of Γ_f span NS_X . The relations among these generators are generated by

$$\sum_{i=1}^{r_v} m_{v,i} C_{v,i} = \sum_{i=1}^{r_{v'}} m_{v',i} C_{v',i} \quad (v, v' \in R_f).$$

Therefore the condition (Z1) implies that the cohomology classes of the irreducible components of Δ constitute a subset of a \mathbb{Z} -basis of NS_X . Hence $\mathbb{Z}[\Delta]$ is primitive in $H^2(X; \mathbb{Z})$. In particular, $\pi_1(X \setminus \Delta)$ is a perfect group by Lemma 4.6. On the other hand, the condition (Z1) implies that there exists a point $v_0 \in \mathbb{P}^1$ such that every fiber of the restriction

$$f|_{X \setminus (\Delta \cup f^{-1}(v_0))} : X \setminus (\Delta \cup f^{-1}(v_0)) \longrightarrow \mathbb{P}^1 \setminus \{v_0\}$$

of f has a reduced irreducible component. Then, by Nori's lemma [13, Lemma 1.5 (C)], if U is a non-empty connected classically open subset of $\mathbb{P}^1 \setminus \{v_0\}$, then the inclusion of $f^{-1}(U) \setminus (f^{-1}(U) \cap \Delta)$ into $X \setminus (\Delta \cup f^{-1}(v_0))$ induces a surjection on the fundamental groups. The inclusion of $X \setminus (\Delta \cup f^{-1}(v_0))$ into $X \setminus \Delta$ also induces a surjection on the fundamental groups. We shall show that there exists a small open disk U on $\mathbb{P}^1 \setminus \{v_0\}$ such that

$$G_U := \pi_1(f^{-1}(U) \setminus (f^{-1}(U) \cap \Delta))$$

is abelian. When (Z2-a) occurs, we take a small open disk disjoint from R_f as U . Then G_U is abelian, because of $f^{-1}(U) \cap \Delta = \emptyset$. Suppose that (Z2-b) occurs. We can take v_0 from $\mathbb{P}^1 \setminus \{v_1\}$, because F_1 has no irreducible components of multiplicity

≥ 2 . We choose as U a small open disk with the center v_1 . There is a contraction from $f^{-1}(U) \setminus (f^{-1}(U) \cap \Delta)$ to $F_1 \setminus (F_1 \cap \Delta)$. Because $\pi_1(F_1 \setminus (F_1 \cap \Delta))$ is abelian, so is G_U . Suppose that (Z2-c) occurs. By Lemma 2.2, there exists a singular fiber $F_2 := f^{-1}(v_2)$ of type I_1 , II , III or IV . Because F_2 has no irreducible components of multiplicity ≥ 2 , we can choose v_0 from $\mathbb{P}^1 \setminus \{v_2\}$. If F_2 is of type I_1 or II , then $F_2 \cap \Delta$ consists of a nonsingular point of F_2 , and $\pi_1(F_2 \setminus (F_2 \cap \Delta))$ is abelian. Hence G_U is also abelian. If F_2 is of type III or IV , then $F_2 \cap \Delta$ cannot coincide with the whole fiber F_2 . Hence Lemma 4.8 implies that G_U is abelian. Therefore we see that $\pi_1(X \setminus \Delta)$ is abelian. Being both perfect and abelian, $\pi_1(X \setminus \Delta)$ is trivial. \square

Now we proceed to the proof of Claim 1. We list up all Σ satisfying the condition (N2) and $\text{rank}(L(\Sigma)) = 18$. It consists of 297 elements. Among them, 199 elements can be the type Σ_f of singular fibers of some extremal elliptic $K3$ surface $f : X \rightarrow \mathbb{P}^1$ with $MW_f = 0$. For these configurations, $\pi_1(X \setminus \Delta)$ is trivial by Claim 2. The remaining 98 configurations are listed in the second column of Table 1 below. Each of them is a sub-configuration of Γ_f satisfying the conditions (Z1) and (Z2), where $f : X \rightarrow \mathbb{P}^1$ is the extremal elliptic $K3$ surface with $MW_f = 0$ whose number in Table 2 is given in the third column of Table 1. The fourth and fifth columns of Table 1 indicate Σ_f and $eu(\Sigma_f)$, respectively. In the case nos. 20, 28, 39, 41 and 85 in Table 1, we can choose the embedding of Δ into Γ_f in such a way that (Z2-b) holds. In the case nos. 30, 37, 57 and 63 in Table 1, we can choose the embedding of Δ into Γ_f in such a way that (Z2-a) holds. By Claim 2 again, $\pi_1(X \setminus \Delta)$ is trivial for these 98 configurations Δ . \square

Remark 4.10. The graph $\Gamma(A_{19})$ (resp. $\Gamma(D_{19})$) can be embedded into Γ_f in such a way that (Z1) and (Z2) are satisfied, where $f : X \rightarrow \mathbb{P}^1$ is the extremal elliptic $K3$ surfaces whose number in Table 2 is 312 (resp. 320). Therefore, if $\Gamma(\Delta)$ is embedded in $\Gamma(A_{19})$ or $\Gamma(D_{19})$, then $\Gamma(\Delta)$ can be embedded in Γ_f in such a way that (Z1) and (Z2) are satisfied.

Table 1. List of embedding of Δ in Γ_f

no	Δ	No	Σ_f	$eu(\Sigma_f)$
1	$A_2 + A_3 + 2 A_4 + A_5$	19	$A_2 + 2 A_3 + A_4 + A_6$	23
2	$A_1 + A_2 + A_3 + 2 A_6$	23	$A_1 + A_2 + A_4 + A_5 + A_6$	23
3	$2 A_1 + A_4 + 2 A_6$	23	$A_1 + A_2 + A_4 + A_5 + A_6$	23
4	$2 A_2 + 2 A_4 + A_6$	23	$A_1 + A_2 + A_4 + A_5 + A_6$	23
5	$A_1 + A_5 + 2 A_6$	40	$A_1 + A_4 + A_6 + A_7$	22
6	$A_4 + 2 A_7$	52	$A_4 + A_6 + A_8$	21
7	$A_1 + A_2 + 2 A_4 + A_7$	23	$A_1 + A_2 + A_4 + A_5 + A_6$	23
8	$A_3 + 2 A_4 + A_7$	24	$A_3 + A_4 + A_5 + A_6$	22
9	$A_2 + 2 A_4 + A_8$	36	$A_2 + A_4 + A_5 + A_7$	22
10	$2 A_3 + A_4 + A_8$	46	$A_1 + A_2 + A_3 + A_4 + A_8$	23
11	$A_3 + A_7 + A_8$	53	$A_1 + A_2 + A_7 + A_8$	22
12	$A_1 + 2 A_2 + A_4 + A_9$	46	$A_1 + A_2 + A_3 + A_4 + A_8$	23
13	$A_2 + A_3 + A_4 + A_9$	71	$2 A_2 + A_4 + A_{10}$	22
14	$A_3 + A_4 + A_{11}$	93	$A_2 + A_4 + A_{12}$	21
15	$A_7 + A_{11}$	312	$A_{10} + E_8$	21
16	$2 A_3 + A_{12}$	93	$A_2 + A_4 + A_{12}$	21
17	$A_3 + A_{15}$	312	$A_{10} + E_8$	21
18	$A_2 + 2 A_6 + D_4$	99	$A_2 + A_3 + A_{13}$	21
19	$2 A_4 + A_6 + D_4$	18	$A_1 + A_3 + 2 A_4 + A_6$	23
20	$2 A_2 + A_4 + A_6 + D_4$	20	$A_1 + 2 A_2 + A_3 + A_4 + A_6$	24
21	$A_2 + A_4 + A_8 + D_4$	44	$2 A_1 + 2 A_4 + A_8$	23
22	$A_6 + A_8 + D_4$	50	$2 A_1 + A_2 + A_6 + A_8$	23
23	$2 A_2 + A_{10} + D_4$	72	$2 A_1 + A_2 + A_4 + A_{10}$	23
24	$A_4 + A_{10} + D_4$	72	$2 A_1 + A_2 + A_4 + A_{10}$	23
25	$A_2 + A_{12} + D_4$	90	$2 A_1 + 2 A_2 + A_{12}$	23
26	$A_{14} + D_4$	320	$D_{10} + E_8$	22
27	$2 A_2 + A_4 + 2 D_5$	210	$2 A_2 + D_{14}$	22
28	$A_1 + 2 A_2 + 2 A_4 + D_5$	157	$A_1 + A_2 + 2 A_4 + D_7$	24
29	$A_2 + A_3 + 2 A_4 + D_5$	46	$A_1 + A_2 + A_3 + A_4 + A_8$	23
30	$A_2 + A_6 + 2 D_5$	193	$A_2 + A_6 + D_{10}$	22
31	$A_3 + A_4 + A_6 + D_5$	18	$A_1 + A_3 + 2 A_4 + A_6$	23
32	$A_2 + A_4 + A_7 + D_5$	72	$2 A_1 + A_2 + A_4 + A_{10}$	23
33	$A_6 + A_7 + D_5$	50	$2 A_1 + A_2 + A_6 + A_8$	23
34	$A_2 + A_3 + A_8 + D_5$	50	$2 A_1 + A_2 + A_6 + A_8$	23
35	$A_3 + A_{10} + D_5$	69	$A_1 + 2 A_2 + A_3 + A_{10}$	23
36	$A_2 + A_{11} + D_5$	90	$2 A_1 + 2 A_2 + A_{12}$	23
37	$A_4 + 2 D_7$	213	$A_4 + D_{14}$	21
38	$A_3 + 2 A_4 + D_7$	44	$2 A_1 + 2 A_4 + A_8$	23
39	$2 A_2 + A_3 + A_4 + D_7$	20	$A_1 + 2 A_2 + A_3 + A_4 + A_6$	24
40	$A_2 + A_4 + A_5 + D_7$	23	$A_1 + A_2 + A_4 + A_5 + A_6$	23
41	$A_1 + 2 A_2 + A_6 + D_7$	14	$2 A_1 + 2 A_2 + 2 A_6$	24

Table 1. List of embedding of Δ in Γ_f

no	Δ	No	Σ_f	$eu(\Sigma_f)$
42	$2 A_2 + A_7 + D_7$	90	$2 A_1 + 2 A_2 + A_{12}$	23
43	$A_4 + A_7 + D_7$	44	$2 A_1 + 2 A_4 + A_8$	23
44	$A_1 + A_2 + A_8 + D_7$	50	$2 A_1 + A_2 + A_6 + A_8$	23
45	$A_3 + A_8 + D_7$	44	$2 A_1 + 2 A_4 + A_8$	23
46	$A_{11} + D_7$	320	$D_{10} + E_8$	22
47	$A_2 + A_4 + D_5 + D_7$	200	$A_2 + A_5 + D_{11}$	22
48	$A_6 + D_5 + D_7$	186	$A_9 + D_9$	21
49	$A_2 + 2 A_4 + D_8$	66	$A_2 + A_7 + A_9$	21
50	$A_4 + A_6 + D_8$	23	$A_1 + A_2 + A_4 + A_5 + A_6$	23
51	$A_2 + A_8 + D_8$	50	$2 A_1 + A_2 + A_6 + A_8$	23
52	$A_{10} + D_8$	320	$D_{10} + E_8$	22
53	$A_1 + 2 A_4 + D_9$	44	$2 A_1 + 2 A_4 + A_8$	23
54	$A_2 + A_3 + A_4 + D_9$	46	$A_1 + A_2 + A_3 + A_4 + A_8$	23
55	$A_3 + A_6 + D_9$	76	$2 A_1 + A_6 + A_{10}$	22
56	$A_2 + A_7 + D_9$	50	$2 A_1 + A_2 + A_6 + A_8$	23
57	$2 A_2 + D_5 + D_9$	210	$2 A_2 + D_{14}$	22
58	$A_2 + D_7 + D_9$	186	$A_9 + D_9$	21
59	$2 A_2 + A_4 + D_{10}$	72	$2 A_1 + A_2 + A_4 + A_{10}$	23
60	$A_3 + A_4 + D_{11}$	44	$2 A_1 + 2 A_4 + A_8$	23
61	$A_7 + D_{11}$	320	$D_{10} + E_8$	22
62	$A_2 + D_5 + D_{11}$	186	$A_9 + D_9$	21
63	$D_7 + D_{11}$	218	D_{18}	20
64	$A_2 + A_4 + D_{12}$	72	$2 A_1 + A_2 + A_4 + A_{10}$	23
65	$A_6 + D_{12}$	320	$D_{10} + E_8$	22
66	$A_1 + 2 A_2 + D_{13}$	90	$2 A_1 + 2 A_2 + A_{12}$	23
67	$A_2 + A_3 + D_{13}$	72	$2 A_1 + A_2 + A_4 + A_{10}$	23
68	$A_3 + D_{15}$	320	$D_{10} + E_8$	22
69	$A_2 + D_{16}$	320	$D_{10} + E_8$	22
70	$2 A_1 + A_4 + 2 E_6$	303	$A_1 + A_4 + A_5 + E_8$	23
71	$2 A_1 + A_2 + 2 A_4 + E_6$	23	$A_1 + A_2 + A_4 + A_5 + A_6$	23
72	$A_2 + 2 A_3 + A_4 + E_6$	46	$A_1 + A_2 + A_3 + A_4 + A_8$	23
73	$2 A_6 + E_6$	37	$A_1 + 2 A_2 + A_6 + A_7$	23
74	$2 A_3 + A_6 + E_6$	41	$A_5 + A_6 + A_7$	21
75	$A_2 + A_3 + A_7 + E_6$	37	$A_1 + 2 A_2 + A_6 + A_7$	23
76	$2 A_4 + D_4 + E_6$	182	$A_4 + A_5 + D_9$	22
77	$A_2 + A_6 + D_4 + E_6$	183	$A_1 + A_2 + A_6 + D_9$	23
78	$A_8 + D_4 + E_6$	186	$A_9 + D_9$	21
79	$A_1 + D_5 + 2 E_6$	320	$D_{10} + E_8$	22
80	$A_2 + 2 D_5 + E_6$	320	$D_{10} + E_8$	22
81	$A_1 + A_2 + A_4 + D_5 + E_6$	193	$A_2 + A_6 + D_{10}$	22
82	$A_2 + A_3 + D_7 + E_6$	200	$A_2 + A_5 + D_{11}$	22

Table 1. List of embedding of Δ in Γ_f

no	Δ	No	Σ_f	$eu(\Sigma_f)$
83	$A_5 + D_7 + E_6$	320	$D_{10} + E_8$	22
84	$A_2 + D_{10} + E_6$	193	$A_2 + A_6 + D_{10}$	22
85	$A_1 + A_2 + 2 A_4 + E_7$	17	$2 A_1 + A_2 + 2 A_4 + A_6$	24
86	$A_3 + 2 A_4 + E_7$	18	$A_1 + A_3 + 2 A_4 + A_6$	23
87	$2 A_2 + D_7 + E_7$	210	$2 A_2 + D_{14}$	22
88	$A_2 + 2 A_4 + E_8$	36	$A_2 + A_4 + A_5 + A_7$	22
89	$2 A_1 + 2 A_2 + A_4 + E_8$	30	$2 A_2 + A_3 + A_4 + A_7$	23
90	$2 A_3 + A_4 + E_8$	24	$A_3 + A_4 + A_5 + A_6$	22
91	$A_3 + A_7 + E_8$	46	$A_1 + A_2 + A_3 + A_4 + A_8$	23
92	$A_2 + A_4 + D_4 + E_8$	182	$A_4 + A_5 + D_9$	22
93	$A_6 + D_4 + E_8$	186	$A_9 + D_9$	21
94	$A_1 + 2 A_2 + D_5 + E_8$	210	$2 A_2 + D_{14}$	22
95	$A_2 + A_3 + D_5 + E_8$	198	$2 A_2 + A_3 + D_{11}$	23
96	$A_3 + D_7 + E_8$	213	$A_4 + D_{14}$	21
97	$A_2 + D_8 + E_8$	210	$2 A_2 + D_{14}$	22
98	$2 A_1 + A_2 + E_6 + E_8$	320	$D_{10} + E_8$	22

Table 2. List of extremal elliptic $K3$ surfaces

No	Σ	MW	a	b	c
1	$6A_3$	$\mathbb{Z}/(4) \times \mathbb{Z}/(4)$	4	0	4
2	$2A_1 + 4A_4$	$\mathbb{Z}/(5)$	10	0	10
3	$2A_2 + 2A_3 + 2A_4$	(0)	60	0	60
4	$3A_1 + 3A_5$	$\mathbb{Z}/(2) \times \mathbb{Z}/(6)$	2	0	6
5	$4A_2 + 2A_5$	$\mathbb{Z}/(3) \times \mathbb{Z}/(3)$	6	0	6
6	$A_3 + 3A_5$	$\mathbb{Z}/(6)$	4	0	6
7	$2A_1 + 2A_3 + 2A_5$	$\mathbb{Z}/(2) \times \mathbb{Z}/(2)$	12	0	12
8	$A_1 + 2A_2 + A_3 + 2A_5$	$\mathbb{Z}/(6)$	6	0	12
9	$2A_4 + 2A_5$	(0)	30	0	30
10	$2A_2 + A_4 + 2A_5$	$\mathbb{Z}/(3)$	6	0	30
11	$A_1 + A_3 + A_4 + 2A_5$	$\mathbb{Z}/(2)$	12	0	30
12	$A_1 + A_2 + 2A_3 + A_4 + A_5$	$\mathbb{Z}/(2)$	24	12	36
13	$3A_6$	$\mathbb{Z}/(7)$	2	1	4
14	$2A_1 + 2A_2 + 2A_6$	(0)	42	0	42
15	$2A_3 + 2A_6$	(0)	28	0	28
16	$A_2 + A_4 + 2A_6$	(0)	28	7	28
17	$2A_1 + A_2 + 2A_4 + A_6$	(0)	50	20	50
18	$A_1 + A_3 + 2A_4 + A_6$	(0)	10	0	140
			20	0	70
19	$A_2 + 2A_3 + A_4 + A_6$	(0)	24	12	76
20	$A_1 + 2A_2 + A_3 + A_4 + A_6$	(0)	30	0	84
21	$2A_1 + 2A_5 + A_6$	$\mathbb{Z}/(2)$	12	6	24
22	$A_1 + 2A_3 + A_5 + A_6$	$\mathbb{Z}/(2)$	4	0	84
23	$A_1 + A_2 + A_4 + A_5 + A_6$	(0)	30	0	42
			18	6	72
24	$A_3 + A_4 + A_5 + A_6$	(0)	12	0	70
25	$4A_1 + 2A_7$	$\mathbb{Z}/(2) \times \mathbb{Z}/(4)$	4	0	4
26	$2A_2 + 2A_7$	(0)	24	0	24
		$\mathbb{Z}/(2)$	12	0	12
27	$A_1 + A_3 + 2A_7$	$\mathbb{Z}/(8)$	2	0	4
28	$2A_1 + 3A_3 + A_7$	$\mathbb{Z}/(2) \times \mathbb{Z}/(4)$	4	0	8
29	$A_2 + 3A_3 + A_7$	$\mathbb{Z}/(4)$	4	0	24
30	$2A_2 + A_3 + A_4 + A_7$	(0)	12	0	120
31	$2A_1 + A_2 + A_3 + A_4 + A_7$	$\mathbb{Z}/(2)$	20	0	24
32	$A_1 + 2A_5 + A_7$	$\mathbb{Z}/(2)$	6	0	24
33	$3A_1 + A_3 + A_5 + A_7$	$\mathbb{Z}/(2) \times \mathbb{Z}/(2)$	8	0	12

Table 2. List of extremal elliptic $K3$ surfaces

No	Σ	MW	a	b	c
34	$A_1 + A_2 + A_3 + A_5 + A_7$	$\mathbb{Z}/(2)$	12	0	24
35	$2A_1 + A_4 + A_5 + A_7$	$\mathbb{Z}/(2)$	2	0	120
36	$A_2 + A_4 + A_5 + A_7$	(0)	6	0	120
			24	0	30
37	$A_1 + 2A_2 + A_6 + A_7$	(0)	24	0	42
38	$2A_1 + A_3 + A_6 + A_7$	$\mathbb{Z}/(2)$	12	4	20
39	$A_2 + A_3 + A_6 + A_7$	(0)	4	0	168
40	$A_1 + A_4 + A_6 + A_7$	(0)	2	0	280
			18	4	32
41	$A_5 + A_6 + A_7$	(0)	16	4	22
42	$2A_1 + 2A_8$	(0)	18	0	18
		$\mathbb{Z}/(3)$	4	2	10
43	$A_1 + 3A_2 + A_3 + A_8$	$\mathbb{Z}/(3)$	12	0	18
44	$2A_1 + 2A_4 + A_8$	(0)	20	10	50
45	$3A_2 + A_4 + A_8$	$\mathbb{Z}/(3)$	12	3	12
46	$A_1 + A_2 + A_3 + A_4 + A_8$	(0)	6	0	180
47	$A_1 + 2A_2 + A_5 + A_8$	$\mathbb{Z}/(3)$	6	0	18
48	$A_2 + A_3 + A_5 + A_8$	$\mathbb{Z}/(3)$	4	0	18
49	$A_1 + A_4 + A_5 + A_8$	(0)	18	0	30
50	$2A_1 + A_2 + A_6 + A_8$	(0)	18	0	42
51	$A_1 + A_3 + A_6 + A_8$	(0)	10	4	52
52	$A_4 + A_6 + A_8$	(0)	18	9	22
53	$A_1 + A_2 + A_7 + A_8$	(0)	18	0	24
54	$2A_9$	(0)	10	0	10
		$\mathbb{Z}/(5)$	2	0	2
55	$A_1 + A_2 + 2A_3 + A_9$	$\mathbb{Z}/(2)$	4	0	60
56	$2A_1 + 2A_2 + A_3 + A_9$	$\mathbb{Z}/(2)$	6	0	60
57	$A_1 + 2A_4 + A_9$	$\mathbb{Z}/(5)$	2	0	10
58	$3A_1 + A_2 + A_4 + A_9$	$\mathbb{Z}/(2)$	20	10	20
59	$2A_1 + A_3 + A_4 + A_9$	$\mathbb{Z}/(2)$	10	0	20
60	$2A_1 + A_2 + A_5 + A_9$	$\mathbb{Z}/(2)$	12	6	18
61	$A_1 + A_3 + A_5 + A_9$	$\mathbb{Z}/(2)$	10	0	12
62	$A_4 + A_5 + A_9$	(0)	10	0	30
		$\mathbb{Z}/(2)$	10	5	10
63	$3A_1 + A_6 + A_9$	$\mathbb{Z}/(2)$	4	2	36
64	$A_1 + A_2 + A_6 + A_9$	(0)	10	0	42

Table 2. List of extremal elliptic $K3$ surfaces

No	Σ	MW	a	b	c
65	$A_3 + A_6 + A_9$	(0)	2	0	140
66	$A_2 + A_7 + A_9$	(0)	10	0	24
67	$A_1 + A_8 + A_9$	(0)	10	0	18
68	$A_2 + 2 A_3 + A_{10}$	(0)	24	12	28
69	$A_1 + 2 A_2 + A_3 + A_{10}$	(0)	12	0	66
70	$2 A_4 + A_{10}$	(0)	10	5	30
71	$2 A_2 + A_4 + A_{10}$	(0)	6	3	84
			24	9	24
72	$2 A_1 + A_2 + A_4 + A_{10}$	(0)	2	0	330
73	$A_1 + A_3 + A_4 + A_{10}$	(0)	20	0	22
			12	4	38
74	$A_1 + A_2 + A_5 + A_{10}$	(0)	6	0	66
			18	6	24
75	$A_3 + A_5 + A_{10}$	(0)	4	0	66
			12	0	22
76	$2 A_1 + A_6 + A_{10}$	(0)	12	2	26
77	$A_2 + A_6 + A_{10}$	(0)	4	1	58
			16	5	16
78	$A_1 + A_7 + A_{10}$	(0)	2	0	88
			10	2	18
79	$A_8 + A_{10}$	(0)	10	1	10
80	$A_1 + 3 A_2 + A_{11}$	$\mathbb{Z}/(3)$	6	0	12
81	$3 A_1 + 2 A_2 + A_{11}$	$\mathbb{Z}/(6)$	2	0	12
82	$A_1 + 2 A_3 + A_{11}$	$\mathbb{Z}/(4)$	4	0	6
83	$2 A_2 + A_3 + A_{11}$	$\mathbb{Z}/(3)$	4	0	12
		$\mathbb{Z}/(6)$	4	2	4
84	$2 A_1 + A_2 + A_3 + A_{11}$	$\mathbb{Z}/(4)$	6	0	6
		$\mathbb{Z}/(2)$	12	0	12
85	$3 A_1 + A_4 + A_{11}$	$\mathbb{Z}/(2)$	6	0	20
86	$A_1 + A_2 + A_4 + A_{11}$	(0)	12	0	30
87	$2 A_1 + A_5 + A_{11}$	$\mathbb{Z}/(2)$	6	0	12
		$\mathbb{Z}/(6)$	2	0	4
88	$A_2 + A_5 + A_{11}$	$\mathbb{Z}/(3)$	4	0	6
89	$A_1 + A_6 + A_{11}$	(0)	4	0	42
90	$2 A_1 + 2 A_2 + A_{12}$	(0)	12	6	42
91	$A_1 + A_2 + A_3 + A_{12}$	(0)	6	0	52

Table 2. List of extremal elliptic $K3$ surfaces

No	Σ	MW	a	b	c
92	$2A_1 + A_4 + A_{12}$	(0)	2	0	130
			18	8	18
93	$A_2 + A_4 + A_{12}$	(0)	6	3	34
94	$A_1 + A_5 + A_{12}$	(0)	10	2	16
95	$A_6 + A_{12}$	(0)	2	1	46
96	$A_1 + 2A_2 + A_{13}$	(0)	6	0	42
		$\mathbb{Z}/(2)$	6	3	12
97	$3A_1 + A_2 + A_{13}$	$\mathbb{Z}/(2)$	2	0	42
98	$2A_1 + A_3 + A_{13}$	$\mathbb{Z}/(2)$	6	2	10
99	$A_2 + A_3 + A_{13}$	(0)	4	0	42
100	$A_1 + A_4 + A_{13}$	(0)	2	0	70
			8	2	18
		$\mathbb{Z}/(2)$	2	1	18
101	$A_5 + A_{13}$	(0)	4	2	22
102	$2A_2 + A_{14}$	$\mathbb{Z}/(3)$	4	1	4
103	$2A_1 + A_2 + A_{14}$	(0)	12	6	18
		$\mathbb{Z}/(3)$	2	0	10
104	$A_1 + A_3 + A_{14}$	(0)	10	0	12
105	$A_4 + A_{14}$	(0)	10	5	10
106	$3A_1 + A_{15}$	$\mathbb{Z}/(4)$	2	0	4
107	$A_1 + A_2 + A_{15}$	(0)	10	2	10
		$\mathbb{Z}/(2)$	4	0	6
108	$A_3 + A_{15}$	$\mathbb{Z}/(4)$	2	0	2
109	$2A_1 + A_{16}$	(0)	2	0	34
			4	2	18
110	$A_2 + A_{16}$	(0)	6	3	10
111	$A_1 + A_{17}$	(0)	4	2	10
		$\mathbb{Z}/(3)$	2	0	2
112	A_{18}	(0)	2	1	10
113	$2A_4 + 2D_5$	(0)	20	0	20
114	$A_3 + 2A_5 + D_5$	$\mathbb{Z}/(2)$	12	0	12
115	$2A_4 + A_5 + D_5$	(0)	20	0	30
116	$A_1 + A_3 + A_4 + A_5 + D_5$	$\mathbb{Z}/(2)$	12	0	20
117	$A_1 + 2A_6 + D_5$	(0)	14	0	28
118	$2A_2 + A_3 + A_6 + D_5$	(0)	12	0	84
119	$A_1 + A_2 + A_4 + A_6 + D_5$	(0)	20	0	42

Table 2. List of extremal elliptic $K3$ surfaces

No	Σ	MW	a	b	c
120	$A_2 + A_5 + A_6 + D_5$	(0)	6	0	84
			12	0	42
121	$A_1 + A_7 + 2 D_5$	$\mathbb{Z}/(4)$	2	0	8
122	$A_1 + A_2 + A_3 + A_7 + D_5$	$\mathbb{Z}/(4)$	6	0	8
123	$2 A_1 + A_4 + A_7 + D_5$	$\mathbb{Z}/(2)$	8	0	20
124	$A_8 + 2 D_5$	(0)	8	4	20
125	$A_1 + A_4 + A_8 + D_5$	(0)	2	0	180
			18	0	20
126	$A_5 + A_8 + D_5$	(0)	12	0	18
127	$2 A_2 + A_9 + D_5$	(0)	6	0	60
128	$2 A_1 + A_2 + A_9 + D_5$	$\mathbb{Z}/(2)$	2	0	60
129	$A_1 + A_3 + A_9 + D_5$	$\mathbb{Z}/(2)$	8	4	12
130	$A_4 + A_9 + D_5$	(0)	10	0	20
131	$A_1 + A_2 + A_{10} + D_5$	(0)	14	4	20
132	$2 A_1 + A_{11} + D_5$	$\mathbb{Z}/(4)$	2	0	6
133	$A_2 + A_{11} + D_5$	$\mathbb{Z}/(2)$	6	0	6
134	$A_1 + A_{12} + D_5$	(0)	2	0	52
			6	2	18
135	$A_{13} + D_5$	(0)	6	2	10
136	$3 D_6$	$\mathbb{Z}/(2) \times \mathbb{Z}/(2)$	2	0	2
137	$2 A_3 + 2 D_6$	$\mathbb{Z}/(2) \times \mathbb{Z}/(2)$	4	0	4
138	$2 A_2 + 2 A_4 + D_6$	(0)	30	0	30
139	$2 A_1 + 2 A_5 + D_6$	$\mathbb{Z}/(2) \times \mathbb{Z}/(2)$	6	0	6
140	$A_1 + 2 A_3 + A_5 + D_6$	$\mathbb{Z}/(2) \times \mathbb{Z}/(2)$	4	0	12
141	$A_3 + A_4 + A_5 + D_6$	$\mathbb{Z}/(2)$	4	0	30
142	$2 A_6 + D_6$	(0)	14	0	14
143	$A_2 + A_4 + A_6 + D_6$	(0)	6	0	70
144	$A_1 + 2 A_2 + A_7 + D_6$	$\mathbb{Z}/(2)$	6	0	24
145	$A_2 + A_3 + A_7 + D_6$	$\mathbb{Z}/(2)$	4	0	24
146	$A_1 + A_4 + A_7 + D_6$	$\mathbb{Z}/(2)$	6	2	14
147	$A_4 + A_8 + D_6$	(0)	4	2	46
148	$A_1 + A_2 + A_9 + D_6$	$\mathbb{Z}/(2)$	6	0	10
		$\mathbb{Z}/(2)$	4	2	16
149	$A_3 + A_9 + D_6$	$\mathbb{Z}/(2)$	4	0	10
150	$A_2 + A_{10} + D_6$	(0)	6	0	22
151	$A_1 + A_{11} + D_6$	$\mathbb{Z}/(2)$	4	0	6

Table 2. List of extremal elliptic $K3$ surfaces

No	Σ	MW	a	b	c
152	$A_{12} + D_6$	(0)	4	2	14
153	$A_2 + A_5 + D_5 + D_6$	$\mathbb{Z}/(2)$	6	0	12
154	$A_7 + D_5 + D_6$	$\mathbb{Z}/(2)$	4	0	8
155	$2 A_2 + 2 D_7$	(0)	12	0	12
156	$A_2 + 3 A_3 + D_7$	$\mathbb{Z}/(4)$	8	4	8
157	$A_1 + A_2 + 2 A_4 + D_7$	(0)	10	0	60
158	$A_2 + A_3 + A_6 + D_7$	(0)	8	4	44
159	$A_1 + A_4 + A_6 + D_7$	(0)	4	0	70
160	$A_5 + A_6 + D_7$	(0)	2	0	84
161	$2 A_1 + A_2 + A_7 + D_7$	$\mathbb{Z}/(2)$	4	0	24
162	$A_1 + A_3 + A_7 + D_7$	$\mathbb{Z}/(4)$	2	0	8
163	$2 A_1 + A_9 + D_7$	$\mathbb{Z}/(2)$	4	0	10
164	$A_2 + A_9 + D_7$	(0)	2	0	60
165	$A_1 + A_{10} + D_7$	(0)	4	0	22
166	$A_{11} + D_7$	$\mathbb{Z}/(4)$	2	1	2
167	$A_1 + A_5 + D_5 + D_7$	$\mathbb{Z}/(2)$	4	0	12
168	$A_5 + D_6 + D_7$	$\mathbb{Z}/(2)$	2	0	12
169	$2 A_1 + 2 D_8$	$\mathbb{Z}/(2) \times \mathbb{Z}/(2)$	2	0	2
170	$2 A_2 + 2 A_3 + D_8$	$\mathbb{Z}/(2)$	12	0	12
171	$2 A_5 + D_8$	$\mathbb{Z}/(2)$	6	0	6
172	$2 A_1 + A_3 + A_5 + D_8$	$\mathbb{Z}/(2) \times \mathbb{Z}/(2)$	2	0	12
173	$A_1 + A_4 + A_5 + D_8$	$\mathbb{Z}/(2)$	2	0	30
174	$2 A_2 + A_6 + D_8$	(0)	12	6	24
175	$A_1 + A_2 + A_7 + D_8$	$\mathbb{Z}/(2)$	2	0	24
176	$A_1 + A_9 + D_8$	$\mathbb{Z}/(2)$	2	0	10
177	$2 D_5 + D_8$	$\mathbb{Z}/(2)$	4	0	4
178	$A_1 + A_3 + D_6 + D_8$	$\mathbb{Z}/(2) \times \mathbb{Z}/(2)$	2	0	4
179	$2 D_9$	(0)	4	0	4
180	$A_1 + 2 A_2 + A_4 + D_9$	(0)	12	0	30
181	$A_1 + A_3 + A_5 + D_9$	$\mathbb{Z}/(2)$	4	0	12
182	$A_4 + A_5 + D_9$	(0)	4	0	30
183	$A_1 + A_2 + A_6 + D_9$	(0)	4	0	42
184	$2 A_1 + A_7 + D_9$	$\mathbb{Z}/(2)$	4	0	8
185	$A_1 + A_8 + D_9$	(0)	4	0	18
186	$A_9 + D_9$	(0)	4	0	10
187	$A_4 + D_5 + D_9$	(0)	4	0	20

Table 2. List of extremal elliptic $K3$ surfaces

No	Σ	MW	a	b	c
188	$2A_1 + 2A_3 + D_{10}$	$\mathbb{Z}/(2) \times \mathbb{Z}/(2)$	4	0	4
189	$2A_4 + D_{10}$	(0)	10	0	10
190	$A_1 + A_3 + A_4 + D_{10}$	$\mathbb{Z}/(2)$	2	0	20
191	$3A_1 + A_5 + D_{10}$	$\mathbb{Z}/(2) \times \mathbb{Z}/(2)$	4	2	4
192	$A_3 + A_5 + D_{10}$	$\mathbb{Z}/(2)$	2	0	12
193	$A_2 + A_6 + D_{10}$	(0)	2	0	42
194	$A_8 + D_{10}$	(0)	2	0	18
195	$A_1 + A_2 + D_5 + D_{10}$	$\mathbb{Z}/(2)$	4	0	6
196	$A_2 + D_6 + D_{10}$	$\mathbb{Z}/(2)$	2	0	6
197	$A_1 + D_7 + D_{10}$	$\mathbb{Z}/(2)$	2	0	4
198	$2A_2 + A_3 + D_{11}$	(0)	12	0	12
199	$A_1 + A_2 + A_4 + D_{11}$	(0)	6	0	20
200	$A_2 + A_5 + D_{11}$	(0)	6	0	12
201	$A_1 + A_6 + D_{11}$	(0)	6	2	10
202	$2A_1 + 2A_2 + D_{12}$	$\mathbb{Z}/(2)$	6	0	6
203	$A_1 + A_2 + A_3 + D_{12}$	$\mathbb{Z}/(2)$	4	0	6
204	$2A_1 + A_4 + D_{12}$	$\mathbb{Z}/(2)$	4	2	6
205	$A_1 + D_5 + D_{12}$	$\mathbb{Z}/(2)$	2	0	4
206	$D_6 + D_{12}$	$\mathbb{Z}/(2)$	2	0	2
207	$A_1 + A_4 + D_{13}$	(0)	2	0	20
208	$A_5 + D_{13}$	(0)	2	0	12
209	$D_5 + D_{13}$	(0)	4	0	4
210	$2A_2 + D_{14}$	(0)	6	0	6
211	$2A_1 + A_2 + D_{14}$	$\mathbb{Z}/(2)$	2	0	6
212	$A_1 + A_3 + D_{14}$	$\mathbb{Z}/(2)$	2	0	4
213	$A_4 + D_{14}$	(0)	4	2	6
214	$A_1 + A_2 + D_{15}$	(0)	4	0	6
215	$2A_1 + D_{16}$	$\mathbb{Z}/(2)$	2	0	2
216	$A_2 + D_{16}$	$\mathbb{Z}/(2)$	2	1	2
217	$A_1 + D_{17}$	(0)	2	0	4
218	D_{18}	(0)	2	0	2
219	$3E_6$	$\mathbb{Z}/(3)$	2	1	2
220	$2A_3 + 2E_6$	(0)	12	0	12
221	$A_1 + A_3 + 2A_4 + E_6$	(0)	20	0	30
222	$A_1 + A_5 + 2E_6$	$\mathbb{Z}/(3)$	2	0	6
223	$A_2 + 2A_5 + E_6$	$\mathbb{Z}/(3)$	6	0	6

Table 2. List of extremal elliptic $K3$ surfaces

No	Σ	MW	a	b	c
224	$2A_2 + A_3 + A_5 + E_6$	$\mathbb{Z}/(3)$	6	0	12
225	$A_3 + A_4 + A_5 + E_6$	(0)	12	0	30
226	$A_6 + 2E_6$	(0)	6	3	12
227	$A_1 + A_2 + A_3 + A_6 + E_6$	(0)	6	0	84
			12	0	42
228	$2A_1 + A_4 + A_6 + E_6$	(0)	20	10	26
229	$A_2 + A_4 + A_6 + E_6$	(0)	18	3	18
230	$A_1 + A_5 + A_6 + E_6$	(0)	6	0	42
231	$A_1 + A_4 + A_7 + E_6$	(0)	2	0	120
232	$A_5 + A_7 + E_6$	(0)	6	0	24
233	$2A_2 + A_8 + E_6$	$\mathbb{Z}/(3)$	6	3	6
234	$2A_1 + A_2 + A_8 + E_6$	$\mathbb{Z}/(3)$	2	0	18
235	$A_1 + A_3 + A_8 + E_6$	(0)	12	0	18
236	$A_4 + A_8 + E_6$	(0)	12	3	12
237	$A_1 + A_2 + A_9 + E_6$	(0)	12	6	18
238	$A_3 + A_9 + E_6$	(0)	10	0	12
239	$2A_1 + A_{10} + E_6$	(0)	2	0	66
240	$A_2 + A_{10} + E_6$	(0)	6	3	18
241	$A_1 + A_{11} + E_6$	(0)	6	0	12
		$\mathbb{Z}/(3)$	2	0	4
242	$A_{12} + E_6$	(0)	4	1	10
243	$A_3 + A_4 + D_5 + E_6$	(0)	12	0	20
244	$A_1 + A_6 + D_5 + E_6$	(0)	2	0	84
245	$A_7 + D_5 + E_6$	(0)	8	0	12
246	$D_6 + 2E_6$	(0)	6	0	6
247	$A_2 + A_4 + D_6 + E_6$	(0)	6	0	30
248	$A_6 + D_6 + E_6$	(0)	4	2	22
249	$A_1 + A_4 + D_7 + E_6$	(0)	4	0	30
250	$D_5 + D_7 + E_6$	(0)	4	0	12
251	$A_4 + D_8 + E_6$	(0)	8	2	8
252	$A_1 + A_2 + D_9 + E_6$	(0)	6	0	12
253	$A_3 + D_9 + E_6$	(0)	4	0	12
254	$A_1 + D_{11} + E_6$	(0)	2	0	12
255	$D_{12} + E_6$	(0)	4	2	4
256	$2A_2 + 2E_7$	(0)	6	0	6
257	$A_1 + A_3 + 2E_7$	$\mathbb{Z}/(2)$	2	0	4

Table 2. List of extremal elliptic $K3$ surfaces

No	Σ	MW	a	b	c
258	$A_4 + 2 E_7$	(0)	4	2	6
259	$A_1 + 2 A_3 + A_4 + E_7$	$\mathbb{Z}/(2)$	4	0	20
260	$2 A_2 + A_3 + A_4 + E_7$	(0)	12	0	30
261	$2 A_3 + A_5 + E_7$	$\mathbb{Z}/(2)$	4	0	12
262	$A_1 + A_2 + A_3 + A_5 + E_7$	$\mathbb{Z}/(2)$	6	0	12
263	$2 A_1 + A_4 + A_5 + E_7$	$\mathbb{Z}/(2)$	8	2	8
264	$A_2 + A_4 + A_5 + E_7$	(0)	6	0	30
265	$A_1 + 2 A_2 + A_6 + E_7$	(0)	6	0	42
266	$A_2 + A_3 + A_6 + E_7$	(0)	4	0	42
267	$A_1 + A_4 + A_6 + E_7$	(0)	2	0	70
			8	2	18
268	$A_5 + A_6 + E_7$	(0)	4	2	22
269	$2 A_2 + A_7 + E_7$	(0)	6	0	24
270	$2 A_1 + A_2 + A_7 + E_7$	$\mathbb{Z}/(2)$	2	0	24
271	$A_1 + A_3 + A_7 + E_7$	$\mathbb{Z}/(2)$	4	0	8
272	$A_4 + A_7 + E_7$	(0)	6	2	14
273	$A_1 + A_2 + A_8 + E_7$	(0)	6	0	18
274	$A_3 + A_8 + E_7$	(0)	4	0	18
275	$2 A_1 + A_9 + E_7$	$\mathbb{Z}/(2)$	2	0	10
276	$A_2 + A_9 + E_7$	(0)	6	0	10
		$\mathbb{Z}/(2)$	4	1	4
277	$A_1 + A_{10} + E_7$	(0)	2	0	22
			6	2	8
278	$A_{11} + E_7$	(0)	4	0	6
279	$D_4 + 2 E_7$	$\mathbb{Z}/(2)$	2	0	2
280	$A_2 + A_4 + D_5 + E_7$	(0)	6	0	20
281	$A_1 + A_5 + D_5 + E_7$	$\mathbb{Z}/(2)$	2	0	12
282	$A_6 + D_5 + E_7$	(0)	6	2	10
283	$A_2 + A_3 + D_6 + E_7$	$\mathbb{Z}/(2)$	4	0	6
284	$A_5 + D_6 + E_7$	$\mathbb{Z}/(2)$	4	2	4
285	$D_5 + D_6 + E_7$	$\mathbb{Z}/(2)$	2	0	4
286	$A_1 + A_3 + D_7 + E_7$	$\mathbb{Z}/(2)$	4	0	4
287	$A_4 + D_7 + E_7$	(0)	2	0	20
288	$A_1 + A_2 + D_8 + E_7$	$\mathbb{Z}/(2)$	2	0	6
289	$A_2 + D_9 + E_7$	(0)	4	0	6
290	$A_1 + D_{10} + E_7$	$\mathbb{Z}/(2)$	2	0	2

Table 2. List of extremal elliptic $K3$ surfaces

No	Σ	MW	a	b	c
291	$D_{11} + E_7$	(0)	2	0	4
292	$A_2 + A_3 + E_6 + E_7$	(0)	6	0	12
293	$A_1 + A_4 + E_6 + E_7$	(0)	2	0	30
294	$A_5 + E_6 + E_7$	(0)	6	0	6
295	$D_5 + E_6 + E_7$	(0)	2	0	12
296	$2 A_1 + 2 E_8$	(0)	2	0	2
297	$A_2 + 2 E_8$	(0)	2	1	2
298	$2 A_2 + 2 A_3 + E_8$	(0)	12	0	12
299	$2 A_1 + 2 A_4 + E_8$	(0)	10	0	10
300	$A_1 + A_2 + A_3 + A_4 + E_8$	(0)	6	0	20
301	$2 A_5 + E_8$	(0)	6	0	6
302	$A_2 + A_3 + A_5 + E_8$	(0)	6	0	12
303	$A_1 + A_4 + A_5 + E_8$	(0)	2	0	30
304	$2 A_2 + A_6 + E_8$	(0)	6	3	12
305	$2 A_1 + A_2 + A_6 + E_8$	(0)	2	0	42
306	$A_1 + A_3 + A_6 + E_8$	(0)	6	2	10
307	$A_4 + A_6 + E_8$	(0)	2	1	18
308	$A_1 + A_2 + A_7 + E_8$	(0)	2	0	24
309	$2 A_1 + A_8 + E_8$	(0)	2	0	18
310	$A_2 + A_8 + E_8$	(0)	6	3	6
311	$A_1 + A_9 + E_8$	(0)	2	0	10
312	$A_{10} + E_8$	(0)	2	1	6
313	$2 D_5 + E_8$	(0)	4	0	4
314	$A_1 + A_4 + D_5 + E_8$	(0)	2	0	20
315	$A_5 + D_5 + E_8$	(0)	2	0	12
316	$2 A_2 + D_6 + E_8$	(0)	6	0	6
317	$A_4 + D_6 + E_8$	(0)	4	2	6
318	$A_1 + A_2 + D_7 + E_8$	(0)	4	0	6
319	$A_1 + D_9 + E_8$	(0)	2	0	4
320	$D_{10} + E_8$	(0)	2	0	2
321	$A_1 + A_3 + E_6 + E_8$	(0)	2	0	12
322	$A_4 + E_6 + E_8$	(0)	2	1	8
323	$D_4 + E_6 + E_8$	(0)	4	2	4
324	$A_1 + A_2 + E_7 + E_8$	(0)	2	0	6
325	$A_3 + E_7 + E_8$	(0)	2	0	4

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